The semi-classical energy of the Chodos-Thorn string

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Abstract

We compute semi-classical corrections to the energy of rotating Nambu-Goto strings with masses at the ends, using methods from quantum field theory on curved space-times. In the limit of large angular momentum, we recover the Regge intercept $a = 1 + \frac{D-2}{24}$ for D dimensional target space previously found for the open and closed string.

1 Introduction

Recently, it was shown [1, 2] that the semi-classical Regge intercept of both open and closed Nambu-Goto strings in D dimensional target space is

$$a = 1 + \frac{D-2}{24}, \tag{1}$$

a somewhat surprising result, as it is well known [3,4] that the consistency of the covariant quantization scheme requires $a \leq 1$, which is always violated by the above.

Open Nambu-Goto strings can also be seen as a phenomenological model for QCD vortex lines, i.e., for the description of mesons (or, more generally, hadrons). From meson spectroscopy, one obtains values for a which are considerably smaller than $a = 1 + \frac{1}{12}$, cf. [5] for a recent analysis. However, the simple Nambu-Goto string model neglects many features that should be relevant in mesons, such as charge and spin of the valence quarks that the vortex line is supposed to connect. Furthermore, in the range of spins for which meson data are available, it seems difficult to distinguish an intercept from mass or geodesic curvature terms at the boundaries.

In the present work, we investigate the effect of attaching mass points to the ends of the string, in order to model the quark masses. This model has been first proposed by Chodos and Thorn [6], but its quantum properties have not been worked out yet. Here, we perform a semi-classical analysis, i.e., we consider perturbations around classical solutions, truncate the action at second order in the perturbation, and quantize the resulting free theory. The perturbations can be naturally interpreted as fields living on the classical world-sheet, with equations of motion governed by the induced metric and the second fundamental form of the embedding. It thus seems natural to use methods from quantum field theory on curved space-time [7,8] for the renormalization of the free world-sheet Hamiltonian H^0. This is also consistent with the effective string theory framework developed in [9]. The correspondence between the world-sheet Hamiltonian and the target space energy then gives corrections to the classical Regge trajectories. In the limit of large angular momenta we recover the shift in the energy given by (1).

An important aspect in the calculation [1] of the Regge intercept of the open string was the absence of a certain mode (the planar $n = 1$ mode in the terminology used there), which one may naively expect to be present. From the present paper, the absence of this mode can be easily understood: For finite masses at the boundaries, there always is a such mode of frequency equal to the rotation frequency of the classical solution. However, it degenerates in the symplectic form, indicating the presence of a corresponding linearly growing solution. These solutions are in fact Nambu-Goldstone modes for the broken translation invariance in the plane of rotation. The identification with center-of-mass (cm) position and momentum operators can be explicitly checked. The crucial point is that these modes enter the Hamiltonian in the form $\frac{1}{2}P^2 - L^{\mathrm{cm}}_{1,2}$, the first term being the usual kinetic energy due to the cm momentum, the second one the cm angular momentum in the plane of rotation. Both are irrelevant for the determination of the rest mass, i.e., the intercept. Interpreting the open string as the massless limit of the Chodos-Thorn string, one thus sees that the absence of the $n = 1$ mode in [1] is justified, in the sense that also in the massive case, this mode does not contribute to the ground state energy.

For the rotating open string, it was found [1] that the energy density diverges in a non-integrable way towards the boundaries. This divergence had to be cured by a boundary counterterm. At present, there does not seem to a fully developed framework for such boundary renormalizations. In the case of mass points at the ends, there is no such non-integrable divergence of the energy density. However, the energy of the mass point at the boundary requires renormalization. The ambiguity in this renormalization corresponds to a geodesic curvature boundary counterterm. This was precisely the counterterm used in [1], justifying the renormalization procedure followed there.

The article is structured as follows: In the next section, we discuss the classical solutions around which we intend to do perturbation theory. The fluctuations around the classical solutions and their quantization is discussed in Section 3. In Section 4 we discuss the relation between target space energy and the world-sheet Hamiltonian and present our method to compute the expectation value of the latter. We obtain a regularized expression for the expectation value of the free Hamiltonian. As we were not able to evaluate it analytically, the actual computation is performed numerically. Details are discussed in Section 5. We conclude with a summary. Some calculations that were omitted in Section 4 can be found in the appendix.

2 The classical rotating string

The action for the Nambu-Goto string with masses at the ends [6] is given by

$$\mathcal{S} = -\gamma \int_\Sigma \sqrt{|g|} - \sum_{c\in\pm} m_c \int_{\partial_c\Sigma} \sqrt{|h|}, \tag{2}$$

where Σ is the world-sheet, $\partial_\pm\Sigma$ its two boundary components, γ is the string tension, $m_\pm$ the masses at the two boundaries. Furthermore, g is the induced metric in the bulk and h the induced metric on the boundary. We work in signature $(-,+)$.

Following [10], it is convenient to parametrize the rotating string solution as

$$\bar{X}(\tau,\sigma) = R(\tau, \cos\tau \sin\sigma, \sin\tau \sin\sigma, 0), \tag{3}$$

where $\sigma \in [-S_-, S_+]$, $S_\pm < \pi/2$. For simplicity, we here assumed that the target space-time is four dimensional. Adding further dimensions (or deleting one) is straightforward.

(3) is a solution to the above action, provided that

$$\frac{\gamma R}{m_\pm} = \frac{\tan S_\pm}{\cos S_\pm}. \tag{4}$$

The induced metric on the world-sheet and on the boundary, in the coordinates introduced above, is

$$\bar{g}_{\mu\nu} = R^2 \cos^2 \sigma \eta_{\mu\nu}, \qquad \bar{h} = -R^2 \cos^2 \sigma. \tag{5}$$

The bulk metric has scalar curvature

$$\mathcal{R} = -\frac{2}{R^2 \cos^4 \sigma} \tag{6}$$

and the boundary component c the geodesic curvature

$$\kappa_c = \frac{\tan S_c}{R \cos S_c}. \tag{7}$$

The (angular) momenta corresponding to the action (2) are given by

$$\begin{aligned} P^i &= \int \frac{\delta \mathcal{S}}{\delta \partial_0 X_i} \mathrm{d}\sigma \qquad (8) \\ &= -\gamma \int_{-S_-}^{S_+} \sqrt{g} g^{0\nu} \partial_\nu X^i \mathrm{d}\sigma + \sum_c m_c |h|^{-\frac{1}{2}} \partial_0 X^i |_c, \\ L_{ij} &= \int \frac{\delta \mathcal{S}}{\delta \partial_0 X^j} X_i \mathrm{d}\sigma - i \leftrightarrow j \qquad (9) \\ &= \gamma \int_{-S_-}^{S_+} \sqrt{g} g^{0\nu} X_j \partial_\nu X_i \mathrm{d}\sigma - \sum_c m_c |h|^{-\frac{1}{2}} X_j \partial_0 X_i |_c - i \leftrightarrow j. \end{aligned}$$

Here $\cdot|_c$ denotes the evaluation at $\sigma = c S_c$. The target space energy is given by $E = P^0$.

For the energy $\bar{E}$ and the angular momentum $\bar{L} = \bar{L}_{1,2}$ of the solution (3), one finds

$$\begin{aligned} \bar{E} &= \sum_{c \in \pm} \left[\gamma R S_c + \frac{m_c}{\cos S_c} \right] = \gamma R \sum_{c \in \pm} \left[S_c + \frac{1}{\tan S_c} \right], \qquad (10) \\ \bar{L} &= \sum_{c \in \pm} \left[\frac{\gamma R^2}{2} \left(S_c - \frac{\sin 2S_c}{2} \right) + m_c R \frac{\sin^2 S_c}{\cos S_c} \right] = \frac{\gamma R^2}{2} \sum_{c \in \pm} \left[S_c - \frac{\sin 2S_c}{2} + \frac{\sin^2 S_c}{\tan S_c} \right]. \qquad (11) \end{aligned}$$

In the massless limit ($m_\pm \to 0$ with R, γ fixed) this reduces to

$$\begin{aligned} \bar{E} &= \pi \gamma R, \\ \bar{L} &= \frac{\pi \gamma R^2}{2}. \end{aligned}$$

In particular, one finds the famous Regge trajectory

$$\bar{E}^2 = 2\pi\gamma \bar{L}.$$

The Regge intercept a is defined as the shift of the trajectory,

$$E^2 = 2\pi\gamma(L - a), \tag{12}$$

possibly up to correction of $\mathcal{O}(L^{-1})$ (which are not present in the covariant quantization scheme).

To discuss the massive case, let us denote the two components of the energy and the angular momentum in (10), (11) by $\bar{E}_\pm$ and $\bar{L}_\pm$. For large R, we have

$$\begin{aligned}\bar{E}_c &= \frac{\pi\gamma}{2}R + \frac{m_c^{\frac{3}{2}}}{3\gamma^{\frac{1}{2}}}R^{-\frac{1}{2}} + \frac{m_c^{\frac{5}{2}}}{20\gamma^{\frac{3}{2}}}R^{-\frac{3}{2}} + \mathcal{O}(R^{-\frac{5}{2}}), \\ \bar{L}_c &= \frac{\pi\gamma}{4}R^2 - \frac{m_c^{\frac{3}{2}}}{3\gamma^{\frac{1}{2}}}R^{\frac{1}{2}} + \frac{3m_c^{\frac{5}{2}}}{20\gamma^{\frac{3}{2}}}R^{-\frac{1}{2}} + \mathcal{O}(R^{-\frac{3}{2}}).\end{aligned} \tag{13}$$

We thus obtain the modified Regge trajectory

$$\bar{E}^2 = 2\pi\gamma\bar{L} + \frac{2^{\frac{1}{4}}4\pi^{\frac{3}{4}}}{3}\gamma^{\frac{1}{4}}\left(m_+^{\frac{3}{2}} + m_-^{\frac{3}{2}}\right)\bar{L}^{\frac{1}{4}} - \frac{2^{\frac{3}{4}}\pi^{\frac{5}{4}}}{10}\gamma^{-\frac{1}{4}}\left(m_+^{\frac{5}{2}} + m_-^{\frac{5}{2}}\right)\bar{L}^{-\frac{1}{4}} + \mathcal{O}(R^{-1}). \tag{14}$$

This gives the next-to-next-to-leading order correction to the Regge trajectory for non-vanishing quark masses. Analogously to (12), we define the Regge intercept a as the $\mathcal{O}(L^0)$ shift of this relation, i.e.,

$$E^2 = 2\pi\gamma(L-a) + CL^{\frac{1}{4}} + \mathcal{O}(L^{-\frac{1}{4}}), \tag{15}$$

with some constant C.

For later convenience it is helpful to note that the inclusion of an Einstein-Hilbert term

$$\mathcal{S}_{EH} = -\frac{\alpha}{2}\int_\Sigma \mathcal{R}\sqrt{|g|} \tag{16}$$

into the action (2), which by the Gauß-Bonnet theorem is equivalent to the addition of geodesic curvature boundary terms, modifies the subleading term in (14), i.e., [11]

$$\bar{E}^2 = 2\pi\gamma\bar{L} - \frac{4\pi^{\frac{3}{4}}}{2^{\frac{1}{4}}3}\gamma^{\frac{1}{4}}\left[\sum_{c\in\pm}\left(\sqrt{m_c^2+4\alpha\gamma} - 2m_c\right)\sqrt{m_c + \sqrt{m_c^2+4\alpha\gamma}}\right]\bar{L}^{\frac{1}{4}} + \mathcal{O}(L^{-\frac{1}{4}}). \tag{17}$$

Hence, the leading order effects of an Einstein-Hilbert (or geodesic curvature) term and masses at the endpoints occur at the same order. In particular, the coefficient of the subleading term has no definite sign. Furthermore, even for coinciding masses, the coefficient of the $\mathcal{O}(L^{\frac{1}{4}})$ term does not determine the coefficient of the $\mathcal{O}(L^{-\frac{1}{4}})$ term.

3 Quantum fluctuations of the string

Our goal is now to perform a (canonical) quantization of the fluctuations φ around the classical background $\bar{X}$, cf. (3), i.e., we consider

$$X = \bar{X} + \gamma^{-\frac{1}{2}}\varphi.$$

At second order in φ, i.e., at $\mathcal{O}(\gamma^0)$, the fluctuations parallel to the world-sheet drop out of the bulk part of the action [9], and analogously, the fluctuations parallel to the boundary drop out of the boundary action, so that it is natural to parameterize the fluctuations as

$$\varphi^a = f_s\begin{pmatrix}0\\0\\0\\1\end{pmatrix} + f_p\begin{pmatrix}\tan\sigma\\ -\sin\tau/\cos\sigma\\ \cos\tau/\cos\sigma\\ 0\end{pmatrix} + f_r\begin{pmatrix}0\\ \cos\tau\\ \sin\tau\\ 0\end{pmatrix}. \tag{18}$$

Here the *scalar* component f_s describes the fluctuations in the direction perpendicular to the plane of rotation, and the *planar* component f_p describes the fluctuations in the plane of rotation (at least approximately for small σ). These components are orthonormal to each other and the bulk world-sheet. The *radial* component f_r is orthonormal to the others and the boundary of the world-sheet. This component is only relevant at the boundary, as is obvious from the action [10]

$$\begin{aligned}\mathcal{S}_0 = &\frac{1}{2}\int_\Sigma \left(\dot{f}_p^2 - {f'_p}^2 - \tfrac{2}{\cos^2\sigma} f_p^2 + \dot{f}_s^2 - {f'_s}^2\right) \mathrm{d}\sigma\mathrm{d}\tau \\ &+ \frac{1}{2}\sum_{c\in\pm} \frac{1}{\tan S_c}\int_{\partial_c\Sigma} \Big(\dot{f}_p^2 + \dot{f}_r^2 + \dot{f}_s^2 + \tfrac{1}{\cos^2 S_c} f_p^2 \\ &\qquad + (1+2\tan^2 S_c) f_r^2 + \tfrac{2}{\cos S_c}(\dot{f}_p f_r - f_p \dot{f}_r)\Big)\,\mathrm{d}\tau. \end{aligned} \tag{19}$$

Obviously, going to higher dimensional target space-time simply amounts to multiplying the number of scalar fields. Furthermore, it should be noted that the string world-sheet is actually curved, cf. (5). This does not matter for the canonical quantization procedure described in this section, but will be important in the discussion of renormalization in the following one.

From the action (19), one obtains the bulk equations of motion (where derivates w.r.t. τ are denoted by dots and those w.r.t. σ by primes)

$$-\ddot{f}_s + f''_s = 0, \tag{20}$$

$$-\ddot{f}_p + f''_p - \tfrac{2}{\cos^2\sigma} f_p = 0, \tag{21}$$

supplemented by the boundary conditions

$$-\ddot{f}_s(\pm S_\pm) = \pm\tan S_\pm f'_s(\pm S_\pm), \tag{22}$$

$$-\ddot{f}_p(\pm S_\pm) + \tfrac{1}{\cos^2 S_\pm} f_p(\pm S_\pm) - \tfrac{2}{\cos S_\pm}\dot{f}_r(\pm S_\pm) = \pm\tan S_\pm f'_p(\pm S_\pm), \tag{23}$$

$$-\ddot{f}_r(\pm S_\pm) + \left(1+2\tan^2 S_\pm\right) f_r(\pm S_\pm) + \tfrac{2}{\cos S_\pm}\dot{f}_p(\pm S_\pm) = 0. \tag{24}$$

In fact these boundary conditions can also be interpreted as equations of motion on the boundary, with boundary values of normal derivatives of the bulk fields as sources (on the r.h.s. of the equations). This point of view was taken in [12], where it was shown that the *scalar sector*, i.e., (20) and (22), has a well-posed initial value formulation and causal propagation. It is obvious that the scalar fluctuations decouple, whereas the planar and radial one are coupled. We thus introduce the notation

$$f_q = (f_p, f_r)$$

for the perturbations in the *planar sector*.

In the massless limit, the boundary conditions (22) for the scalar polarization turn into Neumann boundary conditions, as for the open string [1]. However, the planar boundary conditions do not converge to the boundary conditions $f_p(\pm\frac{\pi}{2}) = 0 = f'_p(\pm\frac{\pi}{2})$ of the open string, cf. also below.

Using the equations of motion (20) – (24), it is straightforward to check that the

symplectic form

$$\sigma((f^1,\dot f^1),(f^2,\dot f^2)) = \int_{-S_-}^{S_+} \left(f_s^1\dot f_s^2 - \dot f_s^1 f_s^2 + f_p^1\dot f_p^2 - \dot f_p^1 f_p^2\right) + \sum_{c\in\pm} \tfrac{1}{\tan S_c}\Big(f_s^1\dot f_s^2 - \dot f_s^1 f_s^2 + f_p^1\dot f_p^2 - \dot f_p^1 f_p^2 + f_r^1\dot f_r^2 - \dot f_r^1 f_r^2 - \tfrac{2}{\cos S_c}(f_r^1 f_p^2 - f_p^1 f_r^2)\Big)$$

is conserved.

We will canonically quantize this system. Due to the presence of single time derivative terms in the action (19), the symplectic form is non-standard. Canonical quantization in such a situation is systematically developed in the appendix to [13]. All our results, in particular on the behavior of symplectically non-normalizable modes, is consistent with the general results derived there.

The basis of the canonical quantization of the system are mode solutions, i.e., solutions of the form

$$f_{s,n}(\tau,\sigma) = f_{s,n}(\sigma)e^{-i\omega_n^s\tau},$$
$$f_{q,n}(\tau,\sigma) = f_{q,n}(\sigma)e^{-i\omega_n^q\tau}.$$

The corresponding modes for the bulk equations of motions are

$$f_{s,n} = A\cos\omega_n^s\sigma + B\sin\omega_n^s\sigma, \tag{25}$$
$$f_{p,n} = A\left(\omega_n^q\cos\omega_n^q\sigma + \tan\sigma\sin\omega_n^q\sigma\right) + B\left(\omega_n^q\sin\omega_n^q\sigma - \tan\sigma\cos\omega_n^q\sigma\right). \tag{26}$$

Setting B (A) to zero yields (anti-) symmetric modes, which are realized for coinciding masses $m_+ = m_-$, by symmetry.

One easily checks that both scalar and planar modes always have the lowest non-negative eigenvalues $\omega_0 = 0$, $\omega_1 = 1$, where

$$f_{s,0} = 1, \qquad f_{q,0} = (\tan\sigma, 0), \tag{27}$$
$$f_{s,1} = \sin\sigma, \qquad f_{q,1} = (\tfrac{1}{\cos\sigma}, i).$$

These have a natural geometric interpretation [10]: The scalar zero mode corresponds to a translation in the direction orthogonal to the plane of rotation and the planar zero mode to a rotation in that plane. The scalar $\omega = 1$ mode corresponds to rotations in a plane spanned by $\vec{e}_3$ and a vector in the plane of rotation, and the planar $\omega = 1$ mode to translations in the plane of rotation.[1] These modes can thus be interpreted as (pseudo-) Goldstone modes for these broken symmetries.

Note that in the planar sector, the modes with odd (even) n are (anti-) symmetric, in contrast to the open string case [1]. This is a manifestation of the fact, discussed above, that the planar boundary conditions of the Chodos-Thorn string do not turn into the boundary conditions of the open string in the massless limit. Nevertheless, in both cases the same intercept a is found.

There are solutions growing linearly in time, associated to the zero modes (27). One

[1] The phase of the mode determines the corresponding vector in the plane of rotation.

easily checks that they form canonical pairs with the zero modes, when normalized as

$$f_{s,Q} = \frac{1}{\sqrt{\sum_{c\in\pm}(S_c + \cot S_c)}} 1,$$

$$f_{s,P} = \frac{1}{\sqrt{\sum_{c\in\pm}(S_c + \cot S_c)}} \tau$$

and

$$f_{q,\theta} = \frac{1}{\sqrt{\sum_{c\in\pm}(S_c + \frac{\sin 2S_c}{1+\sin^2 S_c})}} (\tan\sigma, 0),$$

$$f_{q,\lambda} = -\frac{1}{\sqrt{\sum_{c\in\pm}(S_c + \frac{\sin 2S_c}{1+\sin^2 S_c})}} (\tau\tan\sigma, \mp\tfrac{2\sin S_\pm}{1+\sin^2 S_\pm}),$$

i.e.,[2]

$$\sigma(f_{s,Q}, f_{s,P}) = \sigma(f_{p,\theta}, f_{p,\lambda}) = 1.$$

We note the unusual sign of the linearly growing mode $f_{q,\lambda}$, which was also found for the analogous f_λ mode in the closed string [2]. It is natural to interpret $(f_{s,Q}, f_{s,P})$, or rather the coefficients of these modes, as a pair of position and momentum perpendicular to the plane of rotation and $(f_{q,\theta}, f_{q,\lambda})$ as a pair of angle and angular momentum in the $1-2$ plane. This will be corroborated below.

For the planar sector, there is even a linearly growing solution associated to the $n = 1$ mode. To be precise, we define

$$f_{q,Q} = \frac{1}{\sqrt{2\sum_{c\in\pm}(S_c + \cot S_c)}} (\tfrac{1}{\cos\sigma}, i)e^{-i\tau},$$

$$f_{q,P} = \frac{1}{\sqrt{2\sum_{c\in\pm}(S_c + \cot S_c)}} (\tfrac{\tau}{\cos\sigma} + i\cos\sigma, i\tau - 1)e^{-i\tau} + u f_{q,Q},$$

with

$$u = -\frac{i}{4}\frac{\sum_{c\in\pm}(3S_c + 4\cot S_c) - \cos(S_+ - S_-)\sin(S_+ + S_-)}{\sum_{c\in\pm}(S_c + \cot S_c)}.$$

We then have

$$\sigma(\overline{f_{q,Q}}, f_{q,P}) = 1, \qquad \sigma(f_{q,Q}, f_{q,P}) = \sigma(\overline{f_{q,Q}}, f_{q,Q}) = \sigma(\overline{f_{q,P}}, f_{q,P}) = 0. \tag{28}$$

Hence, $(\overline{f_{q,Q}}, f_{q,P})$ and $(f_{q,Q}, \overline{f_{q,P}})$ are pairs of canonically conjugate variables. The linearly growing modes $f_{q,P}$, $\overline{f_{q,P}}$ correspond to a uniform movement in the plane of rotation. This suggest that we should view these modes as positions and momenta in the plane of rotation. This is further corroborated below.

The scalar modes with $n \geq 1$ and the planar modes with $n \geq 2$ are normalized symplectically as

$$\sigma(\overline{f_{r,n}}, f_{r',n'}) = -i\delta_{rr'}\delta_{nn'}, \tag{29}$$

[2]Here and in the following, we identify a solution f with its Cauchy data $(f, \dot{f})$.

where $r \in \{s, q\}$. As obvious from (28), this is not possible for the planar $n = 1$ mode. The normalization (29) amounts to

$$\delta_{nm} = (\omega_n^s + \omega_m^s) \left[\int_{-S_-}^{S_+} f_{s,n} f_{s,m} + \sum_{c \in \pm} \tfrac{1}{\tan S_c} f_{s,n} f_{s,m} \right], \tag{30}$$

$$\begin{aligned}\delta_{nm} = (\omega_n^q + \omega_m^q) &\left[\int_{-S_-}^{S_+} f_{p,n} f_{p,m} + \sum_{c \in \pm} \tfrac{1}{\tan S_c} \left(f_{p,n} f_{p,m} - f_{r,n} f_{r,m} \right) \right] \\ &+ \sum_{c \in \pm} \tfrac{2i}{\sin S_c} \left(f_{r,n} f_{p,m} + f_{p,n} f_{r,m} \right).\end{aligned} \tag{31}$$

for $n, m > 0$.

In order to prepare for the canonical quantization, we write

$$\phi_s = \sum_{n \in \mathbb{N}_s} (a_{s,n} f_{s,n} + \text{h.c.}) + Q_s f_{s,Q} + P_s f_{s,P} \tag{32}$$

$$\phi_q = \sum_{n \in \mathbb{N}_q} (a_{q,n} f_{q,n} + \text{h.c.}) + \theta f_{q,\theta} + \lambda f_{q,\lambda} + (Q_q f_{q,Q} + P_q f_{q,P} + \text{h.c.}), \tag{33}$$

where

$$\mathbb{N}_s = \{n \geq 1\}, \qquad \mathbb{N}_q = \{n \geq 2\},$$

and the coefficients $Q_s, P_s, \theta, \lambda$ are real. One then finds, for the expansion of the energy, cf. (8),

$$\begin{aligned}E &= \bar{E} + \sqrt{\gamma} \left[\int_{-S_-}^{S_+} \tan \sigma \dot{f}_p(\sigma) \mathrm{d}\sigma + \sum_{c \in \pm} \left(\dot{f}_p(cS_c) + \tfrac{2}{\cos S_c} f_r(cS_c) \right) \right] + \mathcal{O}(\gamma^0), \\ &= \bar{E} + \sqrt{\gamma} \sqrt{\textstyle\sum_{c \in \pm} (S_c + \frac{\sin 2S_c}{1 + \sin^2 S_c})} \sigma(f_{q,\theta}, \phi) + \mathcal{O}(\gamma^0) \\ &= \bar{E} + \sqrt{\gamma} \sqrt{\textstyle\sum_{c \in \pm} (S_c + \frac{\sin 2S_c}{1 + \sin^2 S_c})} \lambda + \mathcal{O}(\gamma^0).\end{aligned} \tag{34}$$

Similarly, one obtains for the angular momentum and the momenta, cf. (8), (9),

$$L_{1,2} = \bar{L}_{1,2} + \sqrt{\gamma} R \sqrt{\textstyle\sum_{c \in \pm} (S_c + \frac{\sin 2S_c}{1 + \sin^2 S_c})} \lambda + \mathcal{O}(\gamma^0), \tag{35}$$

$$P^3 = \sqrt{\gamma \textstyle\sum_{c \in \pm} (S_c + \cot S_c)} P_s + \mathcal{O}(\gamma^0) = \sqrt{\bar{E}/R} P_s + \mathcal{O}(\gamma^0), \tag{36}$$

$$P^1 = \sqrt{2\gamma \textstyle\sum_{c \in \pm} (S_c + \cot S_c)} \Im \bar{P}_q + \mathcal{O}(\gamma^0) = \sqrt{2\bar{E}/R} \Im \bar{P}_q + \mathcal{O}(\gamma^0), \tag{37}$$

$$P^2 = \sqrt{2\gamma \textstyle\sum_{c \in \pm} (S_c + \cot S_c)} \Re P_q + \mathcal{O}(\gamma^0) = \sqrt{2\bar{E}/R} \Re P_q + \mathcal{O}(\gamma^0). \tag{38}$$

This supports the identification of the modes $f_{q,\lambda}$, $f_{s,P}$, $f_{q,P}$ with (angular) momenta discussed above.

Canonical quantization now proceeds as follows: One introduces annihilation and creation operators $\hat{a}_{r,n}$, $\hat{a}_{r,n}^*$ for $r \in \{s, q\}$, $n \in \mathbb{N}_r$, fulfilling

$$[\hat{a}_{r,n}, \hat{a}_{r',n'}^*] = \delta_{rr'} \delta_{nn'}.$$

Furthermore, one introduces position operators $\hat{Q}_s, \hat{\theta}, \hat{Q}_q, \hat{Q}^*_q$ and momenta $\hat{P}_s, \hat{\lambda}, \hat{P}_q, \hat{P}^*_q$ with commutation relations[3]

$$[\hat{Q}_s, \hat{P}_s] = i, \qquad [\hat{\theta}, \hat{\lambda}] = i, \qquad [\hat{Q}^*_q, \hat{P}_q] = i, \qquad [\hat{Q}_q, \hat{P}_q] = 0.$$

Then one replaces the coefficients in (32), (33) by the hatted corresponding operators. The fulfillment of the canonical equal time commutation relations then follows from completeness of the modes. Mathematically, this is expressed by the fact that the Cauchy data of

$$\{f_{s,n}, \overline{f_{s,n}}\}_{n\in\mathbb{N}_s} \cup \{f_{s,Q}, f_{s,P}\} \cup \{f_{q,n}, \overline{f_{q,n}}\}_{n\in\mathbb{N}_q} \cup \{f_{q,Q}, f_{q,P}, \overline{f_{q,Q}}, \overline{f_{q,P}}, f_{q,\theta}, f_{q,\lambda}\}$$

are a basis of a Krein space with indefinite inner product given by

$$[f|g] = i\sigma(\bar{f}, g),$$

or, more precisely, that

$$\begin{aligned}\sum_{r\in\{s,q\}} \sum_{n\in\mathbb{N}_r} \left(|f_{r,n}][f_{r,n}| - |\overline{f_{r,n}}][\overline{f_{r,n}}|\right) + i|f_{s,Q}][f_{s,P}| - i|f_{s,P}][f_{s,Q}| + i|f_{q,\theta}][f_{q,\lambda}| - i|f_{q,\lambda}][f_{q,\theta}| \\ + i|f_{q,Q}][f_{q,P}| - i|f_{q,P}][f_{q,Q}| + i|\overline{f_{q,Q}}][\overline{f_{q,P}}| - i|\overline{f_{q,P}}][\overline{f_{q,Q}}| = \mathbb{1}.\end{aligned}$$

This is due to the fact that the Hamiltonian on this Krein space is Krein self-adjoint, definitizable, and regular at infinity [14] and has a real spectrum. As proofs of these statement lie outside of the main interest of this paper, we omit them.

Omitting the positions and momenta (this will be justified below) we thus have quantum fields ϕ_s, ϕ_q with two-point functions

$$\begin{aligned} w_s(x;x') &:= \langle\Omega|\phi_s(x)\phi_s(x')|\Omega\rangle = \sum_{n\in\mathbb{N}_s} f_{s,n}(x)\overline{f_{s,n}}(x'), \qquad (39)\\ w_q(x;x') &:= \langle\Omega|\phi_q(x)\phi_q(x')|\Omega\rangle = \sum_{n\in\mathbb{N}_q} f_{q,n}(x)\overline{f_{q,n}}(x'), \end{aligned}$$

where for the planar sector one has to take into account also the radial component at the boundary.

4 The world-sheet Hamiltonian and the target space energy

The free world-sheet Hamiltonian for the fluctuations around the rotating string solutions has been derived in [10]:

$$\begin{aligned} H^0 &= \frac{1}{2}\int_{-S_-}^{S_+} \left(\dot{\phi}_p^2 + {\phi'_p}^2 + \tfrac{2}{\cos^2\sigma}\phi_p^2 + \dot{\phi}_s^2 + {\phi'_s}^2\right) \mathrm{d}\sigma \\ &\quad + \frac{1}{2}\sum_{c\in\pm} \frac{1}{\tan S_c}\left(\dot{\phi}_p^2 + \dot{\phi}_r^2 - \tfrac{1}{\cos^2 S_c}\phi_p^2 - (1 + 2\tan^2 S_c)\phi_r^2 + \dot{\phi}_s^2\right). \qquad (40) \end{aligned}$$

With (32), (33), this can formally be written as

$$H^0 = \frac{1}{2}\sum_{r\in\{s,q\}}\sum_{n\in\mathbb{N}_r} n\left(\hat{a}_{r,n}\hat{a}^*_{r,n} + \hat{a}^*_{r,n}\hat{a}_{r,n}\right) - \frac{1}{2}\hat{\lambda}^2 + \frac{1}{2}\hat{P}_s^2 + i\hat{P}_q\hat{Q}^*_q - i\hat{P}^*_q\hat{Q}_q + \hat{P}^*_q\hat{P}_q \qquad (41)$$

[3]The complex positions $\hat{Q}_q$ can be represented on $L^2(\mathbb{R}^2)$ as $\hat{Q}_q = \frac{1}{\sqrt{2}}(Q_1 + iQ_2)$, and analogously for the momenta.

To understand the significance of this free world-sheet Hamiltonian, we note the relation

$$H = RE^q - L^q_{1,2} \tag{42}$$

between the full world-sheet Hamiltonian H and the quantum corrections E^q and $L^q_{1,2}$ to the (target space) energy and angular momentum. The latter are defined by the split

$$E = \bar{E} + E^q, \qquad\qquad L_{1,2} = \bar{L}_{1,2} + L^q_{1,2},$$

into the classical and the φ dependent parts. The relation (42) is a consequence of the fact that H generates translations in the world-sheet time τ, whereas E^q generates translations in the target space time X^0. The factor R is due to the relation between the two, cf. (3). Furthermore, the time evolution generated by H acts on the coefficient of the basis vectors v_p, v_r, v_s, cf. (18). The first two of these rotate, which is seen by the time evolution generated by E^q. To correct this, the generator of rotations has to be added. The relation (42) has been already checked to first order in the perturbation, cf. (34) and (35), as the Hamiltonian H does not have a first order term. It can also easily be checked that the second order term on the right hand side coincides with the free Hamiltonian (40).

Furthermore, the classical solution breaks the time translation invariance to discrete translations $X^0 \mapsto X^0 + 2\pi R$. These correspond to world-sheet translations $\tau \mapsto \tau + 2\pi$. Hence,

$$E^q = \tfrac{1}{R} H \mod \tfrac{1}{R}.$$

With (42), it follows that $L^q_{1,2}$ must have spectrum in the integers, as expected for an angular momentum operator. By (35), this implies that $\hat{\lambda}$ has a discrete spectrum with eigenvalue 0. In the following, we are only considering such eigenstates. In particular, this means that the first order corrections to $L_{1,2}$ and E vanish.

Let us thus consider the second order correction to E^2. Using that the first order variation $\delta^1 E$ of E vanishes (we write $E = \bar{E} + \sum_k \delta^k E$, with k denoting the order of the perturbation φ appearing in $\delta^k E$), we have, by (13) and (42),

$$\delta^2 E^2 = 2\bar{E}\delta^2 E = 2\pi\gamma(\delta^2 L_{1,2} + H^0) + \mathcal{O}(R^{-\frac{3}{2}}),$$

where we also used that, by (5), (4) and (8), (9), $R\delta^2 E$ and $\delta^2 L_{1,2}$ are classically of $\mathcal{O}(R^0)$. Plugging this into (14), we find, with $E^2 = \bar{E}^2 + \delta^2 E^2$ and $L = \bar{L} + \delta^2 L$,

$$E^2 = 2\pi\gamma(L + H^0) + \frac{4\pi}{3}\gamma^{\frac{1}{2}} \left(m_+^{\frac{3}{2}} + m_-^{\frac{3}{2}} \right) \left(\frac{2}{\gamma\pi} \right)^{\frac{1}{4}} L^{\frac{1}{4}} + \mathcal{O}(L^{-\frac{1}{4}}). \tag{43}$$

Comparison with (15) shows that we can determine the intercept a by computing the $\mathcal{O}(R^0)$ contribution of the vacuum expectation value of the free Hamiltonian H^0.[4]

Let us discuss the influence of the P^2 terms in (41). Using (36), (37), (38), and (13), we see that the leading order contribution to E^2 from these terms is

$$E^2 = \frac{\pi\gamma R}{\bar{E}} P_i^2 = P_i^2 + \mathcal{O}(R^{-\frac{3}{2}}),$$

as one would expect. For the determination of the intercept, this spatial momentum contribution to the energy should of course be neglected. Furthermore, one can easily see

[4] As we will see below, the expectation value of H^0 has a term of $\mathcal{O}(R^{\frac{1}{2}})$, due to logarithmic divergences. This, however, is a renormalization ambiguity, corresponding to a geodesic curvature boundary term affecting the coefficient of the $\mathcal{O}(L^{\frac{1}{4}})$ term, cf. (17).

that the PQ terms in (41) are the center of mass contribution to the angular momentum $-L_{1,2}$. By (42), such a term has to be expected in H, as, for a non-zero spatial momentum, one can, by a translation, change the angular momentum $L_{1,2}$ without changing the energy. (42) can thus only be correct if this is compensated in H. For the determination of the Regge trajectory, one has of course to consider a vanishing center of mass contribution to the angular momentum. Hence, all but the first term on the r.h.s. of (41) should be neglected for the determination of the intercept.

The first term on the r.h.s. of (41) naturally decomposes into a scalar and a planar contribution,

$$H^0 = H^0_s + H^0_q.$$

Let us first concentrate on the scalar contribution H^0_s. Formally, its vacuum expectation value is given by

$$\langle H^0_s \rangle = \frac{1}{2} \sum_{n \geq 1} \omega^s_n. \tag{44}$$

This sum is of course quadratically divergent. As long as one does not impose some conditions on the renormalization prescription, one can obtain any result. The renormalization prescription that we are going to employ is based on the framework of locally covariant field theory [7], where the renormalization is performed locally, by using the local geometric data. In that framework, the expectation value of Wick squares (possibly with derivatives) is determined as follows:

$$\langle \Omega | (\nabla^\alpha \phi \nabla^\beta \phi)(x) | \Omega \rangle := \lim_{x' \to x} \nabla^\alpha \nabla'^\beta \left(w(x; x') - h(x; x') \right)$$

Here α, β are multiindices, w is the two-point function

$$w(x; x') = \langle \Omega | \phi(x) \phi(x') | \Omega \rangle$$

in the state Ω, and h is a distribution which is covariantly constructed out of the geometric data, the so-called *Hadamard parametrix*. Importantly, for physically reasonable states, the difference $w - h$ is smooth, so that the above coinciding point limit exists and is independent of the direction from which x' approaches x. This method has been reliably used for the computation of Casimir energies and vacuum polarization, cf. [1] for a list of references.

Let us start by considering the bulk. According to (39), the scalar two-point function w_s is given by

$$w_s(\tau, \sigma; \tau', \sigma') = \sum_{n \in \mathbb{N}_1} f_{s,n}(\sigma) f_{s,n}(\sigma') e^{-i\omega^s_n(\tau - \tau')}.$$

Note that, as discussed above, the contribution of the zero mode is suppressed. It is advantageous to perform the coinciding point limit from the time direction, i.e., we take $x' = (\tau + t, \sigma)$, where $x = (\tau, \sigma)$ and $t \to +0$. We then obtain

$$\frac{1}{2}(\partial_0 \partial'_0 + \partial_1 \partial'_1) w_s(x; x') = \frac{1}{2} \sum_{n \in \mathbb{N}_1} (\omega^s_n c^2_n)^2 e^{i\omega^s_n(t + i\varepsilon)},$$

where c^s_n are the normalization constants for the scalar modes (25) such that (30) holds. We also applied the appropriate $i\varepsilon$ prescription to ensure convergence. Using the asymptotic form of ω^s (for $S_+ = S_-$, this was proven in [12])

$$\omega^s_n = \frac{(n-1)\pi}{S_+ + S_-} + \frac{1}{(n-1)\pi} \sum_{c \in \pm} \tan S_c + \mathcal{O}((n-1)^{-3}), \tag{45}$$

one finds

$$d_n^s := (\omega_n^s)^2 c_n^2 = \frac{\pi(n-1)}{(S_+ + S_-)^2} + \mathcal{O}((n-1)^{-3}). \tag{46}$$

For a scalar field with a variable mass $m^2(x)$ in two dimensional space-time, the Hadamard parametrix is given by (see, e.g., [15])

$$h(x;x') = -\frac{1}{4\pi}\left(1 + \frac{1}{2}m^2(x)\rho(x,x') + \mathcal{O}((x-x')^3)\right)\log\frac{\rho_\varepsilon(x,x')}{\Lambda^2}, \tag{47}$$

where ρ is the so-called Synge world function, i.e., $\frac{1}{2}$ times the geodesic distance of x and x' and Λ is a length scale (the "renormalization scale"). For the local covariance, it is crucial that Λ is fixed and does not depend on any geometric data. Inside of the logarithm, the world function is equipped with an $i\varepsilon$ prescription as follows:

$$\rho_\varepsilon(x,x') = \rho(x,x') + i\varepsilon(t-t'),$$

with t some time function. For the scalar part, where $m=0$, we compute

$$\frac{1}{2}(\partial_0\partial_0' + \partial_1\partial_1')h_s = -\frac{1}{8\pi}\left(\frac{\partial_0\partial_0'\rho + \partial_1\partial_1'\rho}{\rho} - \frac{\partial_0\rho\partial_0'\rho + \partial_1\rho\partial_1'\rho}{\rho^2}\right) + \mathcal{O}(x-x').$$

In our point-splitting limit from the time direction, the denominators become $-\frac{1}{2}t^2R^2\cos^2\sigma$ and $\frac{1}{4}t^4R^4\cos^4\sigma$, respectively. In order to expand the numerators, we apply powers of ∇_0 and take the limit of coinciding points. Using standard identities for the coinciding point limit of derivatives of ρ, cf. [16] for example, and (6), we obtain

$$\frac{1}{2}(\partial_0\partial_0' + \partial_1\partial_1')h_s = -\frac{1}{2\pi(t+i\varepsilon)^2} + \frac{1}{12\pi}\frac{1}{\cos^2\sigma} + \mathcal{O}(t).$$

Using

$$\sum_{n=1}^{\infty} n e^{i(n+\frac{b}{n})(t+i\varepsilon)} = -\frac{1}{(t+i\varepsilon)^2} - \frac{1}{12} - b + \mathcal{O}(t), \tag{48}$$

we may thus write

$$\begin{aligned}\frac{1}{2}(\partial_0\partial_0' + \partial_1\partial_1')(w_s - h_s) &= \frac{1}{2}d_1^s e^{i\omega_1^s t} \\ &+ \frac{1}{2}\sum_{n=1}^{\infty}\left[d_{n+1}^s e^{i\omega_{n+1}^s(t+i\varepsilon)} - \frac{\pi n}{(S_+ + S_-)^2} e^{i(\frac{\pi n}{S_+ + S_-} + \frac{\sum \tan S_c}{\pi n})(t+i\varepsilon)}\right] \\ &\qquad - \frac{\pi}{24(S_+ + S_-)^2} - \frac{1}{2\pi(S_+ + S_-)}\sum_c \tan S_c - \frac{1}{12\pi\cos^2\sigma} + \mathcal{O}(t).\end{aligned}$$

From (45), (46) it follows that the sum on the r.h.s. can be bounded uniformly in t and ε. Furthermore, the resulting local bulk energy density is finite. However, one already sees that this is no longer the case in the massless limit $S_\pm \to \frac{\pi}{2}$. Performing the limit of coinciding points and the integration over σ, we thus obtain the bulk contribution to the expectation value of the scalar Hamiltonian:

$$\begin{aligned}\langle H_{s,\mathrm{bk}}^0\rangle = \frac{S_+ + S_-}{2}&\left(d_1^s + \sum_{n=1}^{\infty}\left[d_{n+1}^s - \frac{\pi n}{(S_+ + S_-)^2}\right]\right) \\ &\qquad - \frac{\pi}{24(S_+ + S_-)} - \frac{7}{12\pi}\sum_{c\in\pm}\tan S_c. \end{aligned}\tag{49}$$

For the boundary part, we can not use a 1-dimensional Hadamard parametrix, as the boundary field is not a solution to a free wave equation, cf. the source term on the r.h.s. of (22). The boundary quantum field is in fact a generalized free field [12]. For its renormalization we thus take the following approach: We determine the local singularities and construct a corresponding counterterm out of the local geometric data. Using (45), one straightforwardly obtains

$$|f_{s,n}(\pm S_\pm)|^2 = \frac{(S_+ + S_-)^2 \tan^2 S_\pm}{\pi^3 (n-1)^3} + \mathcal{O}((n-1)^{-5}) \tag{50}$$

for the normalized mode solutions. For the two-point function on the boundary, we thus obtain

$$\begin{aligned} w^{\mathrm{bd}}_{s,\pm}(\tau;\tau') &= \sum_{n\in\mathbb{N}_1} |f_{s,n}(\pm S_\pm)|^2 e^{i\omega^s_n (t+i\varepsilon)} \\ &= i\frac{(S_+ + S_-)\tan^2 S_\pm}{6} t + \frac{\tan^2 S_\pm}{\pi} t^2 \left(\zeta(3) - \frac{3}{4} + \frac{1}{4}\log\frac{-\pi^2 (t+i\varepsilon)^2}{(S_+ + S_-)^2}\right) + \mathcal{O}(t^3), \end{aligned}$$

where $t = \tau' - \tau$. For the corresponding parametrix, we write distances in terms of the local geometric data, i.e., in terms of

$$\rho = \tfrac{1}{2} t^2 R^2 \cos^2 S_\pm,$$

cf. (5), so that a suitable parametrix is

$$h^{\mathrm{bd}}_{s,\pm} = \frac{\tan^2 S_\pm}{2\pi R^2 \cos^2 S_\pm} \rho \log\frac{-\rho_\varepsilon}{\Lambda^2_\pm} + \mathcal{O}(t^3).$$

Here $\Lambda_\pm$ are renormalization length scales which may depend on the boundary component, at least if the masses at the two endpoints are distinguishable.

For the renormalization of the boundary contribution to the Hamiltonian, we thus have to consider

$$\begin{aligned} \partial_0 \partial_0' h^{\mathrm{bd}}_{s,\pm} &= -\frac{\tan^2 S_\pm}{2\pi} \log\frac{(t+i\varepsilon)^2 R^2 \cos^2 S_\pm}{\Lambda^2_\pm} + \mathcal{O}(t) \\ &= \sum_{n=1}^{\infty} \frac{\tan^2 S_\pm}{\pi n} e^{i\frac{\pi n}{S_+ + S_-}(t+i\varepsilon)} - \frac{\tan^2 S_\pm}{2\pi} \log\frac{(S_+ + S_-)^2 R^2 \cos^2 S_\pm}{\pi^2 \Lambda^2_\pm} + \mathcal{O}(t). \end{aligned}$$

The subtraction of the boundary divergences then yields

$$\begin{aligned} \langle H^0_{s,\mathrm{bd},\pm}\rangle &= \lim_{t\to 0} \frac{1}{2\tan S_\pm} \partial_0 \partial_0' (w^{\mathrm{bd}}_\pm - h^{\mathrm{bd}}_\pm) \\ &= \frac{1}{2\tan S_\pm} \left[e^s_{1,\pm} + \sum_{n=1}^{\infty} \left(e^s_{n+1,\pm} - \frac{\tan^2 S_\pm}{\pi n} \right) \right. \\ &\qquad \left. + \frac{\tan^2 S_\pm}{2\pi} \log\frac{(S_+ + S_-)^2 R^2 \cos^2 S_\pm}{\pi^2 \Lambda^2_\pm} \right], \end{aligned}$$

where we used the abbreviation

$$e^s_{n,\pm} := (\omega^s_n)^2 |f_{s,n}(\pm S_\pm)|^2$$

and the expansions (45) and (50).

For the full expectation value of the free scalar Hamiltonian, we thus obtain

$$
\begin{aligned}
\langle H_s^0 \rangle = & \frac{S_+ + S_-}{2} d_1^s + \sum_{c\in\pm} \frac{e^s_{1,c}}{2\tan S_c} \\
& + \sum_{n=1}^{\infty} \left(\frac{S_+ + S_-}{2} d^s_{n+1} + \sum_{c\in\pm} \frac{e^s_{n+1,c}}{2\tan S_c} - \frac{\pi n}{2(S_+ + S_-)} - \sum_{c\in\pm} \frac{\tan S_c}{2\pi n} \right) \\
& - \frac{\pi}{24(S_+ + S_-)} + \sum_{c\in\pm} \frac{\tan S_\pm}{4\pi} \log \frac{(S_+ + S_-)^2 R^2 \cos^2 S_\pm}{\pi^2 \Lambda_\pm^2},
\end{aligned}
$$

where we absorbed the last term in (49) in a change of the scales $\Lambda_\pm$. With integration by parts, and using the equation of motion (20), the boundary condition (22), and the normalization condition (30), one finds

$$
(S_+ + S_-) d_n^s + \sum_{c\in\pm} \frac{1}{\tan S_c} e^s_{n,c} = \omega_n^s,
$$

so that we may write the above as

$$
\begin{aligned}
\langle H_s^0 \rangle = & \frac{1}{2} \left[\omega_1^s + \sum_{n=1}^{\infty} \left(\omega^s_{n+1} - \left(\frac{\pi n}{S_+ + S_-} + \sum_{c\in\pm} \frac{\tan S_c}{n\pi} \right) \right) \right] \\
& - \frac{\pi}{24(S_+ + S_-)} + \sum_{c\in\pm} \frac{\tan S_c}{2\pi} \log \frac{(S_+ + S_-) R \cos S_c}{\Lambda_c}.
\end{aligned}
\tag{51}
$$

In particular, only knowledge of the mode frequencies ω_n^s is required. This expression can thus be seen as the appropriate regularization of (44).

Let us discuss the renormalization ambiguities in our derivation. For this, it is advantageous to write the scalar part of the action in the proper geometric form

$$
\mathcal{S}_0^s = -\frac{1}{2} \int_\Sigma \partial_\mu f_s \partial^\mu f_s \sqrt{|\bar{g}|} \mathrm{d}^2 x - \frac{1}{2} \sum_{c\in\pm} \frac{R \cos S_c}{\tan S_c} \int_{\partial_c \Sigma} \partial_a f_s \partial^a f_s \sqrt{|\bar{h}|} \mathrm{d}x.
$$

In the second term, the roman indices refer to coordinates on the boundary and are raised with $\bar{h}^{-1}$. The factor $\frac{R\cos S_c}{\tan S_c}$ in front of the boundary term should be seen as a coupling constant. Multiplication of a free action with a constant is compensated by the multiplication of the two-point function with the inverse of that constant. It follows that a factor of $\frac{\tan S_c}{R \cos S_c}$ in front of the boundary parametrix is due to this coupling constant. Let us thus consider the corrected boundary parametrix

$$
\tilde{h}^{\mathrm{bd}}_{s,\pm} = \frac{R \cos S_\pm}{\tan S_\pm} h^{\mathrm{bd}}_{s,\pm} = \frac{\kappa_\pm}{2\pi} \rho \log \frac{-\rho_\varepsilon}{\Lambda_\pm^2} + \mathcal{O}(t^3),
$$

where we used the geodesic curvature κ_c, cf. (7). Hence, this parametrix is constructed out of the local geometric data and changing the scale Λ_c amounts to adding a geodesic curvature counterterm at the boundary component c.[5] On the other hand, it is clear that $\tilde{h}^{\mathrm{bd}}_{s,\pm} \mapsto \tilde{h}^{\mathrm{bd}}_{s,\pm} + \lambda_\pm \kappa \rho$ with some coefficients $\lambda_\pm$ is the only locally constructed redefinition of $\tilde{h}^{\mathrm{bd}}_{s,\pm}$ with the correct scaling behavior that contributes to the Hamiltonian. On the other hand, we could also modify the renormalization of the bulk action by adding a locally and

[5]This renormalization freedom has already been noted in [17].

covariantly constructed term to the parametrix (47). The only such modifications with the proper dimension which affect Wick powers with two derivatives are[6]

$$h(x,x') \mapsto h(x,x') + \eta + \lambda \mathcal{R}(x)\rho(x,x') + \xi m^2(x)\rho(x,x') + \mathcal{O}((x-x')^3). \tag{52}$$

In the scalar case, the mass vanishes and also a non-zero λ would not change the scalar contribution to the Hamiltonian. So we have seen that the only renormalization ambiguity for the scalar Hamiltonian amounts to the redefinition

$$\langle H_s^0 \rangle \to \langle H_s^0 \rangle + \sum_{c \in \pm} \lambda_c \tan S_c, \tag{53}$$

corresponding to a geodesic curvature counterterm $\kappa_c \sqrt{|h_c|}$. Note that, by (4), $\tan S_c \sim \sqrt{\gamma R / m_c} \sim L^{\frac{1}{4}}$ for large R, so this is consistent with the fact that an inclusion of geodesic curvature counterterms modifies the Regge trajectory at $\mathcal{O}(L^{\frac{1}{4}})$, cf. (17).

For the planar contribution, $m^2 = -\mathcal{R}$, and a non-zero $\lambda - \xi$ corresponds to adding an Einstein-Hilbert term to the Lagrangian, which by the Gauß-Bonnet theorem is equivalent to adding geodesic curvature terms to the boundary, leading to the same renormalization ambiguity as for the scalar contribution.

Let us note that the final result (51) could also have been obtained by a point-split regularization of the formal expression (44),

$$\langle H_s^0(t) \rangle = \frac{1}{2} \sum_{n \geq 1} \omega_n^s e^{i\omega_n^s (t + i\varepsilon)}$$

combined with a subtraction of the integral over σ of the point-split bulk parametrix and the point-split boundary parametrix. We did not take that approach here, as it is a priori not clear whether the integration over σ commutes with the limit $t \to 0$. For simplicity, we will perform the calculation of the planar contribution in this way. A calculation analogous to the one performed in the scalar case can be found in the appendix.

Analogously to the scalar contribution, the formal expression for the expectation value of the planar contribution is

$$\langle H_q^0 \rangle = \frac{1}{2} \sum_{n \geq 2} \omega_n^q.$$

The point-split version of this is

$$\langle H_q^0(t) \rangle = \frac{1}{2} \sum_{n \geq 2} \omega_n^q e^{i\omega_n^q (t + i\varepsilon)}.$$

To discuss the bulk renormalization with the parametrix, we note that given the metric (5), the mass square which is implicit in (21) is

$$m^2 = \frac{2}{R^2 \cos^4 \sigma}.$$

In our limit of coinciding points from the time directions, we thus obtain[7]

$$\frac{1}{2} \left(\partial_0 \partial_0' + \partial_1 \partial_1' + \tfrac{2}{\cos^2 \sigma} \right) h_q = -\frac{1}{2\pi t^2} - \frac{1}{2\pi \cos^2 \sigma} \log \frac{tR \cos \sigma}{\Lambda} - \frac{1}{24\pi} \frac{1}{\cos^2 \sigma} + \mathcal{O}(t). \tag{54}$$

[6]In principle, also the second fundamental form $\nabla_\mu \partial_\nu \bar{X}^a$ could appear. However, due to the equation of motion $\nabla^\mu \partial_\mu \bar{X}^a = 0$ and the Gauß-Codazzi equation, the Riemannian curvature is the only linearly independent object at the relevant order.

[7]Here and in the following, we omit the $i\varepsilon$ prescription for simplicity. The reader interested in the correct expressions can easily infer them from the analogous calculation for the scalar sector.

Integration over σ yields the following result for the coinciding point divergence due to the bulk:

$$-\frac{S_+ + S_-}{2\pi}\frac{1}{t^2} - \frac{1}{2\pi}\log\frac{tR}{\Lambda}\sum_{c\in\pm}\tan S_c + \frac{S_+ + S_-}{2\pi} - \frac{1}{2\pi}\sum_{c\in\pm}\tan S_c \log(e\cos S_c) - \frac{1}{24\pi}\sum_{c\in\pm}\tan S_c.$$

In particular, the bulk contributes a logarithmic divergence, contrary to the scalar sector. However, its coefficient is $\sum_{c\in\pm}\tan S_c$, so that the renormalization ambiguity due to the bulk (by changing the renormalization scale Λ) is contained in the renormalization ambiguity (53) already determined.[8]

Let us now focus on the boundary contribution. For large n, we have the asymptotic behavior

$$\begin{aligned}
\omega_n^q &= \frac{(n-2)\pi}{S_+ + S_-} + \frac{2}{(n-2)\pi}\sum_{c\in\pm}\tan S_c + \mathcal{O}((n-2)^{-3}), \qquad (55)\\
|f_{p,n}(\pm S_\pm)|^2 &= \frac{(S_+ + S_-)^2\tan^2 S_\pm}{\pi^3(n-2)^3} + \mathcal{O}((n-2)^{-5}),\\
|f_{r,n}(\pm S_\pm)|^2 &= \frac{4(S_+ + S_-)^4\tan^2 S_\pm}{\pi^5(n-2)^5\cos^2 S_\pm} + \mathcal{O}((n-2)^{-7}).
\end{aligned}$$

Hence, considering (40), we expect the following logarithmic singularity in the coinciding point limit at the boundary:

$$-\frac{1}{2}\sum_{c\in\pm}\frac{\tan S_c}{\pi}\log\frac{\pi t}{S_+ + S_-}.$$

As for the scalar contribution, one argues that this divergence should be cancelled by the addition of the counterterm

$$\sum_{c\in\pm}\frac{\tan S_c}{2\pi}\log\frac{tR\cos S_c}{\Lambda_c}.$$

However, as for the scalar contribution, it is advantageous to perform the subtraction in the sum, in such a way that the limit of coinciding points can be commuted with the summation limit. One thus obtains

$$\begin{aligned}
\langle H_q^0\rangle = \frac{1}{2}&\left[\omega_2^q + \sum_{n=1}^{\infty}\left(\omega_{n+2}^q - \frac{n\pi}{S_+ + S_-} - \frac{2}{n\pi}\sum_{c\in\pm}\tan S_c\right)\right] \qquad (56)\\
&- \frac{\pi}{S_+ + S_-}\frac{1}{24} - \frac{S_+ + S_-}{2\pi} + \sum_{c\in\pm}\frac{\tan S_c}{\pi}\log\frac{R(S_+ + S_-)\cos S_c}{\Lambda},
\end{aligned}$$

where once again we absorbed constant multiples of $\tan S_c$ in a redefinition of Λ.

Our attempts to analytically evaluate (51) and (56) failed, so that we resort to numerical calculations.

[8] Also note that a change of the scale Λ corresponds to a redefinition of the parametrix of the form (52).

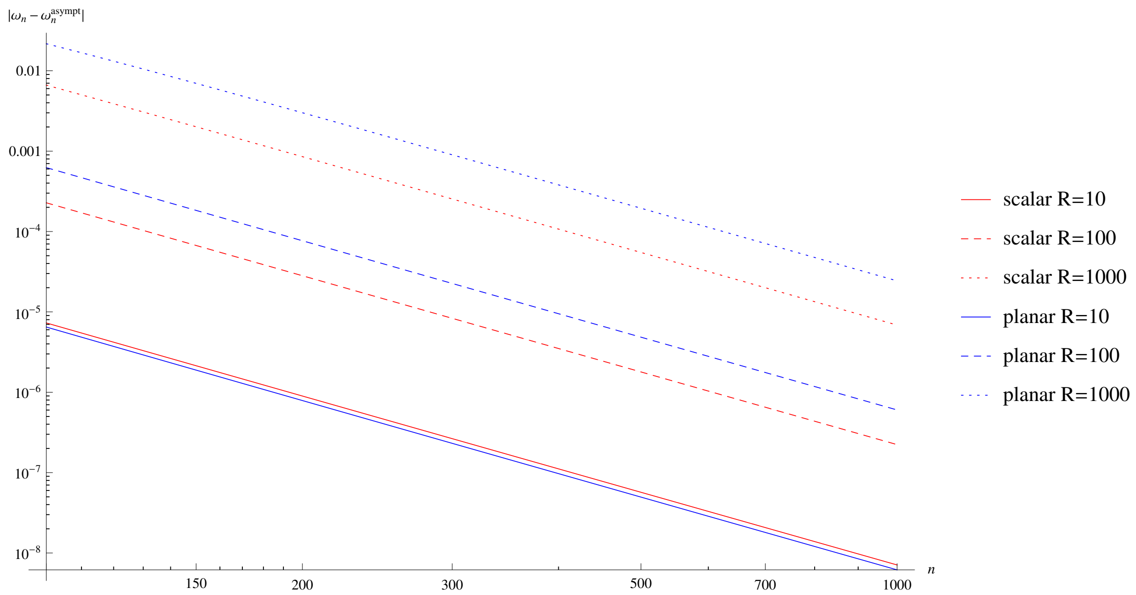

Figure 1: The absolute value of the deviation of the frequencies for $m = 1$, $\gamma = 1$ and different values of R from their asymptotic values as given by (45) and (55).

5 Numerical results

For the numerical calculation, we confine ourselves to the case of equal masses at the endpoints, so in particular $S_+ = S_- = S$. The numerical calculation of (51) and (56) then proceeds as follows:

1. First, we choose a grid of values of R and determine the frequencies ω_n^s, ω_n^q for $n \leq 1000$ by taking the general solutions (25), (26) (with either $B = 0$ or $A = 0$) and looking for zeros of the boundary condition. The absolute value of the deviation of the frequencies from their asymptotic behavior given by (45) and (55) is shown in Figure 1, confirming the estimates (45) and (55).

2. From Figure 1, it is clear that the errors due to a cut-off of the sums in (51) and (56) at some fixed N vary with R. To correct this, we proceed as follows: We choose a grid in N and determine the expressions (51) and (56) for the different values of R, with the sum cut off at N. For fixed R, we fit the result with an $c_0 + c_1 N^{-2}$ ansatz in the range $N \in [500, 1000]$. The number c_0 then gives the result for this R.

3. The resulting function of R is then fitted to
$$C_0 \tan S + C_1 + C_2 R^{-\frac{1}{2}} \tag{57}$$
in the range $R \in [100, 1000]$. The first term corresponds to the renormalization ambiguity and is thus not relevant. The second term, however, directly yields the intercept (up to the sign), according to the discussion below (43). The resulting fits have errors of the order of 2×10^{-7} for the scalar and 5×10^{-7} for the planar part, corresponding to relative errors of the order 5×10^{-8}.

Note that the contribution of the last term in (51) and (56) to the target space energy behaves asymptotically as $R^{-\frac{1}{2}} \log R$, i.e., it slightly dominates the renormalization

ambiguity. The quality of the fits to (57) indicates[9] that it has been properly subtracted, yielding a test of our renormalization prescription.

Our method yields the values

$$C_1^s \simeq -0.04168 \qquad\qquad C_1^q \simeq -1.0417$$

for the scalar and the planar part. These results are quite robust under changes of the fitting range or the fitting function. We interpret them as being the numerical approximation of

$$C_1^s = -\tfrac{1}{24} \simeq -0.04167 \qquad\qquad C_1^q = -1 - \tfrac{1}{24},$$

corresponding to the intercept (1), taking into account that there are $D-3$ scalar polarizations in D dimensional target space. As discussed in [1], the unusual term -1 in C_1^q can be traced to two contributions: The absence of the $n=1$ mode, yielding $-\frac{1}{2}$ and another contribution $-\frac{1}{2}$ stemming from the next to last term in (56), which came from the integration of the logarithmically divergent term in the parametrix (54).

6 Conclusion

Using a locally covariant renormalization scheme, we found the value (1) for the intercept of the Regge trajectory for the Nambu-Goto string with masses at the endpoints, and hence confirmed the result obtained for the massless string [1]. The unusual additional term 1 was interpreted as stemming from two effects: A term $\frac{1}{2}$ is contributed by the absence of the planar $n=1$ mode, for which we gave a geometric explanation. Another term $\frac{1}{2}$ came from the locally covariant renormalization of the bulk energy density, i.e., by the integration over the logarithmically divergent term in the parametrix (54).

Let us comment on the implications of our result for the Nambu-Goto string as a phenomenological model for hadrons. For measured meson trajectories and the endpoint masses and the intercepts as free parameters, intercepts in the range $a \in [-0.55, 0]$ were found [5] (for a fit to the orbital angular momentum), in plain contradiction with our theoretical value $a = 1 + \frac{1}{12}$. However, one has to keep in mind that our semi-classical calculation is only valid for large angular momenta. The maximum spin which was used to determine the trajectories in [5] was $L=6$. But $6^{\frac{1}{4}} \simeq 1.57$, so $L^{\frac{1}{4}}$, L^0 and $L^{-\frac{1}{4}}$ are all of the same order. It seems doubtful that one can consistently distinguish between these contributions with so little data. Apart from that, the model is of course rather crude in that it neglects, for example, the spin of the quarks. However, it is conceivable that fixing a to the theoretical value yields a more consistent assignment of quark masses and the α parameter of the Einstein-Hilbert term (16) to the different trajectories.

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A The calculation of the planar part

In this appendix, we want to discuss the calculation of the planar part in the same fashion as for the scalar part, i.e., without assuming that integration over σ and the limit $t \to 0$ commute. For simplicity, we assume equal masses, i.e., $S_+ = S_- = S$.

[9] One can also include such a term into the fits and finds that it has a very small coefficient.

Let us first concentrate on the bulk. For odd (even) n, the (anti-) symmetric planar mode $f_{p,n}$ is realized. With the normalization given in (26) with $A(B)=1$, we obtain (we set $f_n = f_{p,n}$, $\omega_n = \omega_n^q$)

$$\left(\omega_n^2 + \tfrac{2}{\cos^2\sigma}\right) f_n(\sigma)^2 + f_n'(\sigma)^2 = \omega_n^4 + \omega_n^2 \tan^2\sigma \pm \omega_n^2 \tfrac{2}{\cos^2\sigma} \cos 2\omega_n\sigma \\ \pm 3\omega_n \tfrac{\tan\sigma}{\cos^2\sigma} \sin 2\omega_n\sigma + \tfrac{2\sin^2\sigma+1}{2\cos^4\sigma}(1 \mp \cos 2\omega_n\sigma). \quad (58)$$

Asymptotically, the normalization constants c_n, that have to be multiplied to f_n for the normalization (29), fulfill

$$c_n^2\omega_n^4 = \frac{\pi}{4S^2}(n-2) + \frac{1}{\pi(n-2)} + \mathcal{O}((n-2)^{-3}),$$

so that only the first three terms on the r.h.s. of (58) contribute to the $t \to 0$ singularity of the two-point function. We denote their sum by $T_n(\sigma)$. The remaining terms are denoted by $R_n(\sigma)$. Using (48) and

$$\sum_{n=1}^{\infty} \frac{1}{n} e^{in(t+i\varepsilon)} = -\frac{1}{2}\log -(t+i\varepsilon)^2 + \mathcal{O}(t),$$

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n} e^{in(t+i\varepsilon)} \cos(2n\sigma) = -\frac{1}{2}\log 4\cos^2\sigma + \mathcal{O}(t) \qquad |\sigma| < \tfrac{\pi}{2},$$

we may thus write

$$\begin{aligned}
\tfrac{1}{2}\left(\partial_0\partial_0' + \partial_1\partial_1' + \tfrac{2}{\cos^2\sigma}\right)(w_q - h_q) &= \tfrac{1}{2}c_2^2 T_2(\sigma) \\
&+ \frac{1}{2}\sum_{n=1}^{\infty}\left[c_{n+2}^2 T_{n+2}(\sigma) e^{i\omega_{n+2}(t+i\varepsilon)} - \frac{\pi n}{4S^2} e^{i\left(\frac{\pi n}{2S} + \frac{4\tan S}{\pi n}\right)(t+i\varepsilon)} \right. \\
&\left. - \frac{1}{\pi n\cos^2\sigma}\left(1 - (-1)^n 2\cos 2\tfrac{\pi n}{2S}\sigma\right) e^{i\frac{\pi n}{2S}(t+i\varepsilon)}\right] \\
&+ \frac{1}{2}\sum_{n=2}^{\infty} c_n^2 R_n(\sigma) + \frac{1}{24\pi\cos^2\sigma} - \frac{\pi}{96S^2} - \frac{\tan S}{\pi S} \\
&+ \frac{1}{4\pi\cos^2\sigma}\log\frac{4S^2R^2\cos^2\sigma}{\pi^2\Lambda^2} + \frac{1}{2\pi\cos^2\sigma}\log 4\cos^2\tfrac{\pi}{2S}\sigma.
\end{aligned}$$

In this expression, we may take the limit $t \to 0$, to obtain the bulk energy density

$$\begin{aligned}
\tfrac{1}{2}\left(\partial_0\partial_0' + \partial_1\partial_1' + \tfrac{2}{\cos^2\sigma}\right)(w_q - h_q) &= \tfrac{1}{2}c_2^2 T_2(\sigma) \\
&+ \frac{1}{2}\sum_{n=1}^{\infty}\left[c_{n+2}^2 T_{n+2}(\sigma) - \frac{\pi n}{4S^2} - \frac{1}{\pi n\cos^2\sigma}\left(1 - (-1)^n 2\cos 2\tfrac{\pi n}{2S}\sigma\right)\right] \\
&+ \frac{1}{2}\sum_{n=2}^{\infty} c_n^2 R_n(\sigma) + \frac{1}{24\pi\cos^2\sigma} - \frac{\pi}{96S^2} - \frac{\tan S}{\pi S} \\
&+ \frac{1}{4\pi\cos^2\sigma}\log\frac{4S^2R^2\cos^2\sigma}{\pi^2\Lambda^2} + \frac{1}{2\pi\cos^2\sigma}\log 4\cos^2\tfrac{\pi}{2S}\sigma.
\end{aligned}$$

Due to (58) and the asymptotic forms of c_n and ω_n, the sum can be dominated uniformly in σ for $\sigma \in [-S, S]$ and $S < \frac{\pi}{2}$. Hence, except for the last term, the energy density is bounded for $\sigma \in [-S, S]$ for $S < \frac{\pi}{2}$. The logarithmic divergence of the energy density near

the boundary is a well-known phenomenon in two-dimensional massive scalar field theories, cf. [18] for example. We have thus established that the energy density is integrable and integration over σ yields

$$\begin{aligned}\langle H^0_{q,\mathrm{bk}}\rangle = \tfrac{1}{2}\int c_2^2(T_2(\sigma)+R_2(\sigma))\mathrm{d}\sigma \\ +\frac{1}{2}\sum_{n=1}^{\infty}\left[\int c_{n+2}^2(T_{n+2}(\sigma)+R_{n+2}(\sigma))\mathrm{d}\sigma - \frac{\pi n}{2S} - \frac{2\tan S}{\pi n} + Q_n\right] \\ + \frac{\tan S}{12\pi} - \frac{\pi}{48S} - \frac{2\tan S}{\pi} - \frac{S}{\pi} + \frac{\tan S}{\pi} + \frac{\tan S}{2\pi}\log\frac{4S^2R^2\cos^2\sigma}{\pi^2\Lambda^2} + Q,\end{aligned}$$

with

$$\begin{aligned}Q_n &= (-1)^n\int_{-S}^{S}\frac{2}{\pi n\cos^2\sigma}\cos\tfrac{\pi n}{S}\sigma\mathrm{d}\sigma,\\ Q &= \int_{-S}^{S}\frac{1}{2\pi\cos^2\sigma}\log 4\cos^2\tfrac{\pi}{2S}\sigma\mathrm{d}\sigma.\end{aligned}$$

Using integration by parts, one shows that $|Q_n| < Cn^{-2}$ and

$$\frac{1}{2}\sum_{n=1}^{\infty}Q_n = -Q,$$

so that the above reduces to

$$\begin{aligned}\langle H^0_{q,\mathrm{bk}}\rangle = \frac{1}{2}\int c_2^2(T_2(\sigma)+R_2(\sigma))\mathrm{d}\sigma \\ +\frac{1}{2}\sum_{n=1}^{\infty}\left[\int c_{n+2}^2(T_{n+2}(\sigma)+R_{n+2}(\sigma))\mathrm{d}\sigma - \frac{\pi n}{2S} - \frac{2\tan S}{\pi n}\right] \\ - \frac{\pi}{48S} - \frac{S}{\pi} + \frac{\tan S}{2\pi}\log\frac{4S^2R^2\cos^2\sigma}{\pi^2\Lambda^2},\end{aligned}$$

where we absorbed terms of the form $C\tan S$ in a change of the scale Λ.

For the boundary component, one obtains, analogously to the scalar part,

$$\langle H^0_{q,\mathrm{bd}}\rangle = \frac{1}{\tan S}\left[e_2^q + \sum_{n=1}^{\infty}\left(e_{n+2}^q - \frac{\tan^2 S}{\pi n}\right) + \frac{\tan^2 S}{2\pi}\log\frac{4S^2R^2\cos^2 S}{\pi^2\Lambda^2}\right]$$

where we used

$$e_n^q := c_n^2\left[\left(\omega_n^2 - \tfrac{1}{\cos^2 S}\right)f_{p,n}(S)^2 + \left(\omega_n^2 - 1 - 2\tan^2 S\right)|f_{r,n}|^2\right].$$

In total, we thus have

$$\begin{aligned}\langle H^0_q\rangle = \frac{1}{2}\int c_2^2(T_2(\sigma)+R_2(\sigma))\mathrm{d}\sigma + \frac{1}{\tan S}e_2^q \\ +\frac{1}{2}\sum_{n=1}^{\infty}\left[\int c_{n+2}^2(T_{n+2}(\sigma)+R_{n+2}(\sigma))\mathrm{d}\sigma + \frac{2}{\tan S}e_{n+2}^q - \frac{\pi n}{2S} - \frac{4\tan S}{\pi n}\right] \\ - \frac{\pi}{48S} - \frac{S}{\pi} + \frac{\tan S}{\pi}\log\frac{4S^2R^2\cos^2\sigma}{\pi^2\Lambda^2}.\end{aligned}$$

Using integration by parts, the equations of motion (21), (23), (24), and the normalization (31), one finds

$$\frac{1}{2}\int c_n^2(T_n(\sigma)+R_n(\sigma))\mathrm{d}\sigma + \frac{1}{\tan S}e_n^q = \frac{1}{2}\omega_n^q.$$

Hence, we obtain (56) for the special case $S_+ = S_- = S$.

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